Is Gravitation Mediated by the Torsion of Spacetime?

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We propose a new set of field equations within the framework of a Poincaré gauge theory of gravitation. These equations couple momentum to the torsion and spin to the curvature of spacetime and encompass an Einstein-like limit.

The Poincaré gauge theory of gravitation (Einstein-Cartan-Sciama-Kibble theory, U_4 theory) is based on the following two assumptions:

A.1: *The special relativistic description of material systems, in particular their Poincaré invariance with 4 translational and 6 rotational degrees of freedom, is valid within any sufficiently small region of spacetime.*

A.2: *The relative distances and orientations of the local Minkowski bases are governed by field equations.*

In this way the theory tries to derive the gravitational interaction of elementary fields from the fundamentals of field theory rather than from the behavior of macroscopic bodies.

The assumptions uniquely identify a Riemann-Cartan spacetime as the kinematical arena of the theory. The geometry is described by the constant Minkowski metric $g_{\alpha\beta}$, the 4 translational gauge potentials e_i^{α} (tetrad coefficients) and the 6 rotational gauge potentials $\Gamma_i^{\alpha\beta} = \Gamma_i^{[\alpha\beta]}$ (connection coefficients) *. The corresponding 4-plus-6 gauge fields are the torsion $F_{ij}^{\cdot\cdot\alpha}$ and the curvature $F_{ij}^{\cdot\cdot\alpha\beta}$.

The material Lagrangian density

$$\mathcal{L}(\psi, \partial_i \psi, e_i^{\alpha}, \Gamma_i^{\alpha\beta})$$

is locally identical to the special relativistic density $L(\psi, \partial_\alpha \psi)$, that is $\mathcal{L} = e\, L(\psi, e_\alpha{}^i D_i \psi)$. Provided $\delta\mathcal{L}/\delta\psi = 0$, the 4 translational currents $e\,\Sigma_{\alpha}^{\cdot i} := \delta\mathcal{L}/\delta e_i^{\alpha}$ and the 6 rotational currents $e\,\tau_{\alpha\beta}^{\cdot\cdot i} := \delta\mathcal{L}/\delta\Gamma_i^{\alpha\beta}$ are identical to the canonical energy-momentum and spin angular momentum tensors, respectively, known from special relativity.

As a first step towards a dynamics for the gravitational fields we add a third assumption.

A.3: *The 4-plus-6 field equations for e_i^{α} and $\Gamma_i^{\alpha\beta}$ can be derived from a total Lagrangian density $\mathcal{L} + \mathcal{V}$ with a first order field Lagrangian density $\mathcal{V}(e_i^{\alpha}, \partial_j e_i^{\alpha}, \Gamma_i^{\alpha\beta}, \partial_j \Gamma_i^{\alpha\beta})$.*

The general structure of the field equations is then uniquely determined:

$$D_j \mathcal{H}_{\alpha}^{\cdot ij} + \mathcal{E}_{\alpha}^{\cdot i} = e\, \Sigma_{\alpha}^{\cdot i}\,, \tag{1}$$

$$D_j \mathcal{H}_{\alpha\beta}^{\cdot\cdot ij} + \mathcal{H}_{[\alpha\beta]}^{\cdot\cdot i} = e\, \tau_{\alpha\beta}^{\cdot\cdot i}\,. \tag{2}$$

Here, we have defined the tensor densities $\mathcal{H}_{\alpha}^{\cdot ij} := \partial\mathcal{V}/\partial(\partial_j e_i^{\alpha})$ and $\mathcal{H}_{\alpha i\beta}^{\cdot\cdot j} := \partial\mathcal{V}/\partial(\partial_j \Gamma_i^{\alpha\beta})$. The tensor density $-\mathcal{E}_{\alpha}^{\cdot i} := e_\alpha{}^i \mathcal{V} - F_{\alpha k}^{\cdot\cdot\gamma} \mathcal{H}_{\gamma}^{\cdot ki} - F_{\alpha k}^{\cdot\cdot\gamma\delta} \mathcal{H}_{\gamma\delta}^{\cdot\cdot ki}$ is the covariant part of the energy-momentum supplied by e_i^{α} and $\Gamma_i^{\alpha\beta}$, whereas the tetrad coefficients e_i^{α} alone contribute via $-\mathcal{H}_{[\alpha\beta]}^{\cdot\cdot i}$ to the spin current.

To complete the theory the 4-plus-6 densities $\mathcal{H}$ have to be specified. The rest of the note is devoted to this problem. The usual choice is

Choice 1: $\quad \mathcal{H}_{\alpha}^{\cdot ij} = 0\,; \quad \mathcal{H}_{\alpha\beta}^{\cdot\cdot ij} = (e/l^2)\, e_{[\alpha}^{i}\, e_{\beta]}^{j}\,,$

or equivalently $\mathcal{V} = (e/2\, l^2)\, e_\delta{}^k\, e_\gamma{}^l\, F_{kl}^{\cdot\cdot\gamma\delta}$ where $l \approx 10^{-32}$ cm is the Planck length. The advantages of this choice are maximal simplicity of the field equations

$$F_{\alpha\gamma}^{\cdot\cdot\gamma i} - e_{\alpha}^{i}\, F_{\delta\gamma}^{\cdot\cdot\gamma\delta}/2 = l^2\, \Sigma_{\alpha}^{\cdot i}\,, \tag{3}$$

$$F_{\alpha\beta}^{\cdot\cdot i}/2 + e_{[\alpha}^{i}\, F_{\beta]\gamma}^{\cdot\cdot\gamma} = l^2\, \tau_{\alpha\beta}^{\cdot\cdot i} \tag{4}$$

and their similarity to Einstein's theory. Like Einstein's theory this theory is consistent with present day experiments.

In our opinion, however, the following considerations suggest to look for an alternative choice:

a) The choice 1 introduces an asymmetry between translations and rotations in that $\mathcal{V}$ does not depend on the translational gauge fields, but only on the potentials e_i^{α}.

b) The kinematics of this theory closely resembles that of gauge theories of internal symmetry groups. Choice 1 violates this analogy as far as possible. The field Eqs. (3), (4) are algebraic in the gauge fields and couple these fields to the wrong sources. As a consequence the translational gauge potentials e_i^{α} get coupled to their sources $\Sigma_{\alpha}^{\cdot i}$ only after substitution of (4) into (3).

c) The gauge kinematics suggests that the orbital angular momentum of matter or rather its general relativistic relic, the spin of the tetrad fields, should enter the source in addition to material spin. With choice 1 the tetrad coefficients carry no spin.

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* The quantities and symbols employed in this note are taken from von der Heyde [1]. For more details and literature we refer to the review article by Hehl, von der Heyde, Kerlick, and Nester [2]. Latin indices refer to the coordinate basis and greek indices $\alpha, \beta, \ldots = 0, 1, 2, 3$ number the tetrads. We defined $e := \det(e_i^{\alpha})$ and $e\, e_\alpha{}^i := \partial e/\partial e_i^{\alpha}$. D_i is the operator of parallel translation with respect to Γ. The matter field components ψ are always referred to the tetrads.

d) Momentum and spin are coupled to geometry with the same strength. From the dimensional difference between translations and rotations one might rather expect a length associated with the translational part and a dimensionless coupling constant for the rotational part.

e) It is impossible to retain choice 1 if one wishes to generalize the Poincaré group to the general affine group GA(4,R) including nonmetricity and proper hypermomentum as proposed by Hehl, Kerlick, and von der Heyde[3]. In this case additional terms must enter the Lagrangian $\mathcal{V}$ and the simplicity is lost.

f) Choice 1 does not allow for the usual quantization methods and retains the difficulties known from general relativity in this context.

We propose to examine as an alternative the field equations resulting from the following choice:

$$\textit{Choice 2:} \quad \mathcal{H}_{\alpha}^{\cdot \mathrm{ij}} = (e/l^2)\,(F^{\mathrm{ij}}_{\cdot\cdot\alpha} + 2\, e_{\alpha}^{[\mathrm{i}}\, F^{\mathrm{j}]\gamma}_{\cdot\cdot\gamma})\,,$$
$$\mathcal{H}_{\alpha\beta}^{\cdot\cdot \mathrm{ij}} = (e/\varkappa)\, F^{\mathrm{ij}}_{\cdot\cdot\alpha\beta}\,.$$

Here l is the Planck length and $\varkappa$ is a dimensionless constant to be determined by observation. The corresponding Lagrangian density is

$$\mathcal{V} = (e/4\,l^2)\,(F_{kl}^{\cdot\cdot\gamma}\, F^{\mathrm{lk}}_{\cdot\cdot\gamma} + 2\, F_{k\gamma}^{\cdot\cdot\gamma}\, F^{\mathrm{k}\delta}_{\cdot\cdot\delta}) + (e/4\,\varkappa)\, F_{kl}^{\cdot\cdot\gamma\delta}\, F^{\mathrm{lk}}_{\cdot\cdot\gamma\delta}\,.$$

The field energy-momentum $\mathcal{E}_{\alpha}^{\cdot \mathrm{i}}$ turns out to be asymmetric and traceless.

If the coupling constant $\varkappa$ is sufficiently small, the geometry becomes a teleparallelism to a first approximation. Then, in an appropriate gauge, the connection coefficients Γ approximately vanish and the torsion $F_{ij}^{\cdot\cdot\alpha} = 2\,(\partial_{[i}\, e_{j]}^{\alpha} + \Gamma_{[i|\gamma}^{\cdot\alpha}\; e_{|j]}^{\gamma})$ is essentially given by the first Riemannian term. The density $\mathcal{H}_{\alpha}^{\cdot \mathrm{ij}}$ of choice 2 is the simplest one which gives Newton's law in this limit. A more detailed calculation reveals that *in the limit of teleparallelism* the exact static, spherical symmetric solution of the non-linear Eq. (1) is the Schwarzschild solution. As preliminary computations indicate, deviations from this solution induced by the second field Eq. (2) can be adjusted by the choice of $\varkappa$ in order to check with the classical tests.

The choice of $\mathcal{H}_{\alpha\ \beta}^{\cdot\cdot \mathrm{ij}}$ is strictly analogous to other gauge theories. Furthermore, since $D_i\, D_j\,(e\, F^{\mathrm{ij}}_{\cdot\cdot\alpha\beta}) = 0$ in a Riemann-Cartan spacetime, curvature does not enter the angular momentum theorem $D_i(e\,\tau_{\alpha\beta}^{\cdot\cdot \mathrm{i}} - \mathcal{H}_{[\alpha\beta]}^{\cdot\cdot \mathrm{i}}) = 0$ as should be expected from the noncovariance of the spin of the connection field.

There is additional support for choice 2. Take the special relativistic Dirac theory for the electron and perform Gordon decompositions of the currents Σ and τ. These exactly reflect the structure of the field Eqs. (1), (2) with *non*-vanishing $\mathcal{H}_{\alpha}^{\cdot \mathrm{ij}}$. They lead to choice 2 if one substitutes – in analogy to the electromagnetic case – the moments of the momentum and the spin current by the corresponding gauge fields.

We would like to point out that irrespective of the magnitude of $\varkappa$ the energy-momentum theorem for spinless matter reduces to the one of standard general relativity. This guarantees geodesic motion for macroscopic matter.

Finally we note that the vacuum field equation for gravitation proposed by Yang[4] is contained in (2) in the case of vanishing torsion. In the context of our interpretation, however, there seems to be no good place for this limit as torsion mediates gravity.

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